Lecture 10: What is a Function, definition,
piecewise defined functions, difference quotient, domain of a function

A function arises when one quantity depends on another. Many everyday relationships between variables can be expressed in this form.

Example 1 We have already seen examples of functions in our lecture on lines. For example, if a car leaves Mathville at 1:00 p.m. and travels at a constant speed of 60 miles per hour for two hours, then the distance in miles traveled by the car after t hours(D) (assuming it has not stopped along the way or changed speed) depends on the time, or is a **function** of time (t) = t hours:

$$D = D(t) = 60t \quad \text{for} \quad 0 \leq t \leq 2$$

We can use this general formula to calculate how far the car has traveled at any given time t. For example after 1/2 hour (t = 1/2) the car has traveled a distance of $D = 60\frac{1}{2} = 30$ miles.

Here is another everyday example of a function.

Example Consider the volume of a cylindrical glass with radius = 1 inch. We have a formula for the volume; $V = \pi r^2 h = \pi h \ \ in^3$, where h is the height of the glass in inches and $r = 1$ is the radius.

We see that the value of the volume depends on the height, h; V is a function of h. We sometimes indicate that the value of V depends on the value of h, by writing the formula as $V(h) = \pi h \ \ in^3$.

When $h = 2$, $V = V(2) =$ and when $h = 3, V = V(3) =$

In this section, we will examine the definition and general principles of functions and look at examples of functions which are more complex than lines.

Definition of a Function A **function** is a map or a rule which assigns to each element x of a set A exactly one element called $f(x)$ in a set B.

Example The rule which assigns to each real number its square is a function. We have $A = (-\infty, \infty)$ or the set of all real numbers. Since the square of a real number is a real number, we have B is also the set of real numbers. If x is an element of A, this rule assigns the element $f(x) = x^2$ in B to x. If $x = -2$, the $f(-2) = (-2)^2 = 4$. In short, this is the map or rule which sends x to x^2.

- We will focus on functions where the sets A and B are sets of real numbers. The symbol $f(x)$ is read as "f" of "x" or "f" at "x" and is called the **value of** f **at** x or the **image of** x **under** f.
- The possible values of x vary here since x can take any value from the set A, hence x is a **variable**. We call x **the independent variable**.
- We may replace the symbol x by any other symbol here, for example t or u, in which case the symbol for our function changes to $f(t)$ or $f(u)$ respectively. Likewise we can replace the name of the function, f, by any other name in the definition, for example h or g or D. So if u denotes our independent variable and D is the name of our function, the values of the function are denoted by $D(u)$.

- The set of numbers (or objects) to which we apply the function, A, is called the **domain** of the function.

- The set of values of B which are equal to $f(x)$ for some x in A is called the **range** of f. We have

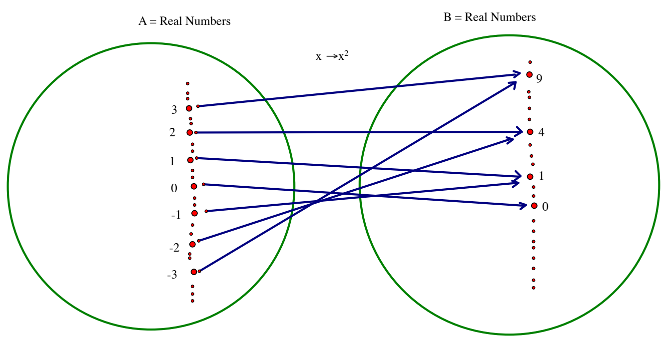
$$\text{range of } f = \{f(x)|x \in A\}$$

  In the example shown above where $f(x) = x^2$, we see that the values of $f(x) = x^2$ are alwys ≥ 0. Furthermore, every positive number is a square of some number, so the range of f here is the set of all real numbers which are ≥ 0.

- If we can write our function with a formula (or a number of formulas), as in the above example where $f(x) = x^2$, then we can represent the function by an equation $y = f(x)$. In our example above, we might just describe our function with the equation $y = x^2$. Here y is also a variable, with values in the range of the function. The value of y depends on the given value of x and hence y is called the **dependent variable**. Note that writing a function in this way allows us to draw a graph of the function in the Cartesian plane.

- It is helpful to think of a function as a machine or process that transforms the number you put through it. If you put in the number x, then the machine or process changes it and gives back $f(x)$.

- Another way to visualize a function is by using an arrow diagram as shown below (for the example $f(x) = x^2$ above).



  For every element, x, in the domain of the function we have **exactly one arrow leaving** the point representing that element, indicating that the function can be applied to any element in the domain and we get exactly one value in B when we apply the function to x.
  On the other hand, **an element in the set B can have 0, 1 or more than one arrow pointing towards it**. If the element is not in the range, it has no arrows arriving at it, if it is in the range it has at least one arrow arriving at it. Note that in our example, we have two arrows arriving at 4 in the set B because the values of f at (-2) and 2 are both 4, i.e. $f(-2) = f(2) = 4$.

Example Lets summarize what we know about our example above where our function is the rule which send any value x from the real numbers to the value x^2.

- As pointed out above, we can give a formula for $f(x)$, namely $f(x) = x^2$. We can use this formula to calculate the value of the function for any given value of x. For example

$$f(-1) = (-1)^2 = 1, \quad f(0) = 0^2 = 0, \quad f(1/2) = (1/2)^2 = 1/4.$$

- The domain is the set of all real numbers and the range is the set $\{y \in \mathbb{R} \mid y \geq 0\}$.
- We can represent this function by an equation $y = x^2$. Here x is the independent variable which can take any values of x in the real numbers (the domain) and y is the dependent variable whose values will be in the range.

Representing a function We saw above that there are a number of ways to represent a function. We can represent it by

1. A verbal description
2. an algebraic representation in the form of a formula for $f(x)$ (possibly not just a single formula).
3. an equation of the form $y = f(x)$, which we can graph on the Cartesian plane
4. an arrow diagram or a table (especially if we have a finite number of points in the domain).

We will focus on using the algebraic description and the graphical description of a function. The Cartesian plane allows us to translate the results we derive algebraically to a graphical interpretation and vice-versa.

Evaluating a function

Here we will focus on deriving and using the algebraic description of some examples of functions.

Example Let $f(x) = x^2 + 2$. Evaluate the following

$$f(-2), \qquad f(0), \qquad f(3/2), \qquad f(10).$$

Piecewise Defined Functions

Not every function can be defined with a single formula (such as the absolute value function), sometimes we may need several lines/formulae to describe a function. These are called **Piecewise Defined Functions**.

Example The cost of (short-term) parking at South Bend airport depends on how long you leave your car in the short term lot. The parking rates are described in the following table:

First 30 minutes	Free
31-60 minutes	\$2
Each additional hour	\$2
24 hour maximum rate	\$13

The Cost of Parking is a function of the amount of time the car spends in the lot. If we are to create a formula for the cost of parking = C, in terms of how long our car stays in the lot = t, we need to give the formula piece by piece as follows:

$$C(t) = \begin{cases} \$0 & 0 \le t \le 0.5\ hr. \\ \$2 & 0.5\ hr. < t \le 1\ hr. \\ \$4 & 1\ hr. < t \le 2\ hr. \\ \$6 & 2\ hr. < t \le 3hr. \\ \$8 & 3\ hr. < t \le 4\ hr. \\ \$10 & 4\ hr. < t \le 5\ hr. \\ \$12 & 5\ hr. < t \le 6\ hr. \\ \$13 & 6\ hr. < t \le 24\ hr. \\ \$13 + \text{cost of towing} & t > 24\ hr. \end{cases}$$

If you were figuring out how much you needed to pay using this description of the function, you would first figure out which category you were in (how long you had parked for) and then note the cost for cars in that category. We proceed in the same way if we are given the formula for any piecewise defined function.

Example Let

$$g(x) = \begin{cases} x+1 & \text{if} \quad x > 1 \\ x^2 & \text{if} \quad -1 \le x \le 1 \\ 4 & \text{if} \quad x < -1 \end{cases}$$

Evaluate the following

$g(-2)$, $g(0)$, $g(3/2)$, $g(10)$.

The Difference Quotient

When we wish to derive general formulas in mathematics, we have to use general variables to represent values in a function so that we can prove a result for all values in the domain. This means that we often have to evaluate the function at some combination of abstract values, such as a, $a+h$, like the ones shown below as opposed to evaluating the function at concrete values such as -1, 0 etc... . Here are some examples of such calculations. The **difference quotient**

$$\frac{f(a+h) - f(a)}{h}$$

is particularly important when learning about derivatives.

Example Let $f(x) = x^2 + 2$ and let a be any real number and h a real number where $h \neq 0$. Evaluate

1. $f(a)$
2. $f(-a)$
3. $f(a^2)$
4. $f(a+h)$
5. $\frac{f(a+h)-f(a)}{h}$

Evaluate $\frac{f(a+h)-f(a)}{h}$ when $a = 2$

Evaluate $\frac{f(a+h)-f(a)}{h}$ when $h = 0.1$

Evaluate $\frac{f(a+h)-f(a)}{h}$ when $a = 2$ and $h = 0.1$.

Example Let $k(x) = 2x+1$. Evaluate $\dfrac{k(a+h)-k(a)}{h}$ where a is any real number and h a real number where $h \neq 0$.

Example Let

$$g(x) = \begin{cases} x+1 & \text{if} \quad x > 1 \\ x^2 & \text{if} \quad -1 \leq x \leq 1 \\ 4 & \text{if} \quad x < -1 \end{cases}$$

(a) Evaluate $\dfrac{g(0+h)-g(0)}{h}$ for values of h for which $|h| < 1$ and $h \neq 0$.

(b) Evaluate $\dfrac{g(1+h)-g(1)}{h}$ for values of h which are greater than 0.

(c) Evaluate $\dfrac{g(1+h)-g(1)}{h}$ for values of h which are greater than -1 and less than 0.

The Domain of A Function

Recall that the domain of a function $f(x)$ is the set of values of x to which we can apply the function. Sometimes we explicitly state what the domain of a function is (see Example A below) and sometimes we just give a formula for the function (see Example B below). In the latter case, it is implicitly assumed that the domain of the given function is the set of all real numbers which make sense in the formula.

Example A Let

$$f(x) = x^3, \qquad 0 \leq x \leq 1$$

Here we have explicitly stated that the domain is the values of x in the interval $[0, 1]$.

Example B Let

$$g(x) = \frac{1}{x-2}$$

Here it is assumed that the domain of this function is all values of x which make sense in the formula, that is the set of all real numbers except 2.

$$\text{Domain of } g = \{x \in \mathbb{R} | x \neq 2\}$$

Note Keep in mind when calculating domains for functions that we <u>cannot have 0 as the denominator of a quotient</u> and that we <u>can only evaluate square roots (or nth roots for n even) for numbers greater than or equal to zero</u>.

Example Find the domain of the following function:

$$f(x) = \sqrt{5 - x^2}$$

Example Find the domain of the following function:

$$R(x) = \sqrt{\frac{x-1}{x-2}}$$

Example Find the domain of the following function:

$$R(x) = \sqrt{\frac{x^2+11}{x^2+3x-10}}$$

Domains of common functions It is good to keep the domains of the following functions in mind:

Function	**Domain**
$x^n, \quad n \in \mathbb{N}$	all $x \in \mathbb{R}$
$\frac{1}{x^n}, \quad n \in \mathbb{N}$	$\{x \in \mathbb{R} \mid x \neq 0\}$
$\sqrt[n]{x}, \quad n \in \mathbb{N}$, n even	$\{x \in \mathbb{R} \mid x \geq 0\}$
$\sqrt[n]{x}, \quad n \in \mathbb{N}$, n odd	all $x \in \mathbb{R}$

Modeling with functions

Example: Income Tax In a certain country, income tax T is assessed according to the following function of income x:

$$T(x) = \begin{cases} 0 & \text{if } 0 \le x \le 10,000 \\ 0.08x & \text{if } 10,000 < x \le 20,000 \\ 1600 + 0.15x & \text{if } 20,000 < x \end{cases}$$

(a) Find $T(5,000)$, $T(12,000)$ and $T(25,000)$.

(b) What do your answers in part (a) represent?